II. Analytic methods in complex algebraic geometry

To continue discussing the question "what is complex algebraic geometry?" we will illustrate how analytic methods (PDEs and differential geometry) lead to what remain as some of the deepest results in the subject. Our emphasis will be on existence and uniqueness theorems, and the techniques illustrated will be restricted to complex dimension 1 but they will extend to the general case. At the end we will return to the integrals of algebraic functions and add a new ingredient. The types of questions to be considered are

- Given a compact Riemann surface (to be defined) and points $p_1, \dots, p_d$ on C, does there exist a meromorphic function w on C having poles on the p_i? How many such functions are there?

Remark that a meromorphic function will be the same as a holomorphic mapping

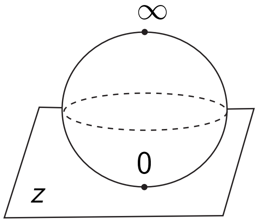
$$w : C \to \mathbb{P}^1$$

from the Riemann surface C to the Riemann surface $\mathbb{P}^1$ ($=$ Riemann sphere).

- Can we construct meromorphic differentials φ on C with poles on the p_i and given residues, subject only to the constraint imposed by the residue theorem $\sum_{p_i} \operatorname{Res} \varphi = 0$?

These are existence results that require analysis to address.

Riemann sphere $\mathbb{C} \cup \{\infty\}$



$$\begin{cases} \mathcal{U}_0 = \mathbb{P}^1 \backslash \{\infty\} \cong \mathbb{C}, & z \\ \mathcal{U}_\infty = \mathbb{P}^1 \backslash \{0\} \cong \mathbb{C}, & z' \\ \mathcal{U}_0 \cap \mathcal{U}_\infty \cong \mathbb{C}^*, & z = \frac{1}{z'}. \end{cases}$$

Linear fractional transformations

$$z \to (az + b)/(cz + d)$$

where $\det \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \neq 0$ act transitively on $\mathbb{P}^1$. Then $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$ sends $0 \to \infty$.

(i) function w on open set is holomorphic or meromorphic if $w(z)$ is holomorphic/meromorphic in the usual sense

(ii) any global holomorphic function is constant (maximum principle — will give another proof below)

(iii) if $w(z)$ is meromorphic on $\mathbb{P}^1$ then

$$\#\text{ zeroes} = \#\text{ poles}$$

Will also recall the proof later.

(iv) there exists $w(z)$ with given zeroes and poles if $\#$ zeroes $= \#$ poles

Proof of (iv): $w(z) = \dfrac{\prod(z-a_i)^{m_i}}{\prod(z-b_j)^{n_j}}$ and $\sum_i m_i = \sum_j n_j$.

(v) w meromorphic on $\mathbb{P}^1$

$$\Downarrow$$

w is rational function

(can work analytically and results will have algebro-geometric meaning)

(vi) $\varphi = w(z)\,dz$ is holomorphic or meromorphic if $w(z)$ is.

The integral $\int_\gamma \varphi$ is well defined for $\gamma =$ path in $\mathbb{P}^1$ not containing any pole of φ.

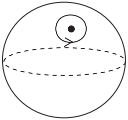
This may be refined to γ not containing any point p with $\operatorname{Res}_p \varphi \neq 0$. Moreover, The integral above will be invariant under continuous deformation of the contour γ provided the deformation does not cross a point p with $\operatorname{Res}_p \varphi \neq 0$.

Example

For $z = \frac{1}{z'}$, $dz = -\frac{dz'}{(z')^2}$. For $w(z) = \sum_{i=0}^{n} a_i z^i$ polynomial

- $w(z)$ has pole of order n at ∞
- $w(z)\,dz$ has pole of order $n+2$ at ∞.

(vii) $\mathrm{Res}_p\,\varphi = \frac{1}{2\pi i}\oint \varphi$,



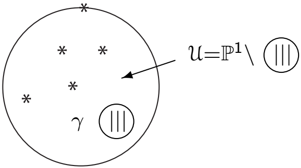
$$\boxed{\sum \mathrm{Res}_p\,\varphi = 0}$$

This implies (iii) where $\varphi = \frac{dw(z)}{w(z)}$.

Example

$\varphi = \frac{dz}{z} = -\frac{dz'}{z'} \implies \mathrm{Res}_0\,\varphi + \mathrm{Res}_\infty\,\varphi = 0$

Proof of (vii):



$$\int_\gamma \varphi = \sum_{p \in \mathcal{U}} \mathrm{Res}_p\, \varphi$$

$$= -\sum_{p \in (|||)} \mathrm{Res}_p\, \varphi = 0$$

(ix) there exist a (unique) φ with given poles and residues $\iff \sum_p \mathrm{Res}_p\, \varphi = 0$

Proof: May assume all $z_i \in \mathbb{C} \subset \mathbb{P}^1$. Then

$$\sum \frac{a_i\, dz}{z - z_i} = -\sum \left(\frac{a_i}{z'(1 - z' z_i)} \right) dz'$$

$$= -\left(\sum a_i\right) \frac{dz'}{z'} + \text{ holomorphic differential near } z' = 0$$

(x) any meromorphic differential φ is

$$\varphi = \varphi' + \varphi''$$

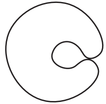
φ': 1st order poles

φ'': no residues

$$\Longrightarrow \int_{z_0}^{z} \varphi = \sum a_i \log(z - z_i) + w(z)$$

$$= \text{elementary function}$$

Variant:[1] Relative algebraic curve $(\mathbb{P}^1; \{0, \infty\})$

Picture =  , require $w(0) = w(\infty)$

There exists $w(z)$ with given zeroes and poles in $\mathbb{P}^1 \backslash \{0, \infty\}$ and $w(0) = w(\infty) \iff$ # zeroes = # poles and

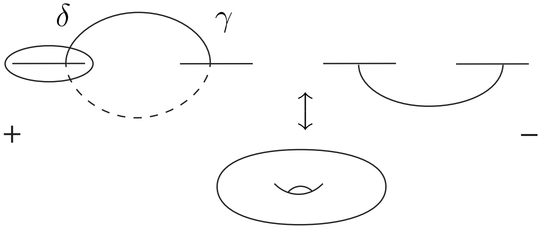
$$\prod a_i^{m_i} = \prod b_j^{n_j}$$

[1] In preparation for lecture # 4

General algebraic curves

For $\mathbb{P}^1$ we can do everything "by hand" using elementary complex analysis. Don't need PDEs, differential geometry, Hodge theory, ... To understand C corresponding to

$$w^2 = p(z), \qquad w = \sqrt{p(x)}$$



the issue is much more subtle and interesting. For example, (v), (ix) become

(v_C) there exist a non-constant meromorphic function w on C with poles at any distinct points $p_1, \ldots, p_d$ if $d \geqq g+1$. For general p_i this is the best possible and if $d = g+1$ the function is unique up to scaling.

(ix_C) there exists φ with given residues $\iff \sum \operatorname{Res} \varphi = 0$. This φ is unique up to $\varphi + \omega$ where $\omega \in H^{1,0}(C)$ has $\int \omega < \infty$ (recall that $h^{1,0}(C) = \dim H^{1,0}(C) = g$)

Results such as these provide the beginnings of

- the Riemann-Roch theorem (dimensions of space of functions in (v_c))
- classification of algebraic curves (moduli; # parameters of C is equal to 1 for $g = 1$ and to $3g - 3$ for $g \geqq 2$)
- Hodge structures and mixed Hodge structures

To prove them requires analysis in the PDE sense, specifically the study on C of the inhomogeneous Cauchy-Riemann equations

$$\overline{\partial} w = \varphi$$

where $w = w(z, \bar{z})$ and $\varphi = v(z, \bar{z})\, d\bar{z}$ are C^∞ in the variables $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$ and

$$\begin{cases} \overline{\partial} w = (\partial_{\bar{z}} w)\, d\bar{z} \\ \partial_{\bar{z}} = \frac{1}{2}(\partial_x - i\partial_y). \end{cases}$$

Then for $w = u + iv$

$$\partial_{\bar{z}} w = 0 \iff \begin{cases} \partial_x u = -\partial_y v \\ \partial_y u = \partial_x v \end{cases}$$

are usual Cauchy-Riemann equations.

We note that for

$$z' = f(z)$$

holomorphic we have the chain rule which we write as

$$f^* \overline{\partial} = \overline{\partial}'$$

(The notations we will use for complex analysis are collected in the appendix to these notes.) Thus to do analysis on an algebraic curve C we need that

(i) C locally looks like an open set in $\mathbb{C}$

(ii) any two such local representations are related by a holomorphic change of variables.

Think of $\mathbb{P}^1$ and $\mathcal{U}_0, \mathcal{U}_\infty$. For our C given by $w^2 = p(z)$

any open set not containing a branch point looks like an open set in $\mathbb{C}$; all we have done is locally choose one branch of $\sqrt{p(z)}$.
At a branch point, say $z = 0$

$$w^2 = zq(z), \qquad q(0) \neq 0.$$

We can take $\sqrt{q(z)}$ and for

$$w' = \frac{w}{\sqrt{q(z)}}$$

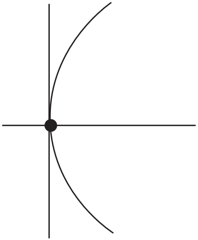
write the equation as

$$w'^2 = z.$$

The mapping

$$t \to (t^2, t)$$
$$\uparrow \; \uparrow$$
$$z \;\; w'$$

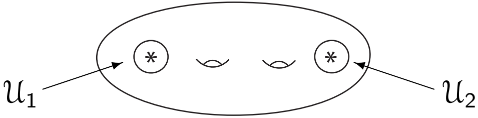
gives an isomorphism between an open set in C and an open set in $\mathbb{C}$. The real picture is



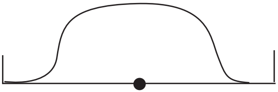
$$y = x^2 + \cdots$$

Construction of global meromorphic functions

Given $p_1, \ldots, p_d \in C$ we want to construct $w : C \to \mathbb{P}^1$ with $w^{-1}(\infty) = p_1 + \cdots + p_d$



- $\rho_i = C^\infty f^{\mathfrak{n}}$ in $\mathcal{U}_i$
  - $= 1$ near p_i
  - $= 0$ on $\partial\mathcal{U}_i$



$$\varphi = \sum_i \overline{\partial}\left(\frac{\rho_i}{z - z_i}\right) \overset{\text{defn}}{=} \sum \frac{\overline{\partial}\rho_i}{z - z_i}$$
$$= C^\infty\ (0,1)\ \text{form on}\ \ C.$$

If we can solve

$$(*) \qquad \overline{\partial}g = \varphi, \quad g = C^\infty f^{\mathfrak{n}}\ \text{ on }\ C$$

then

$$\overline{\partial}\underbrace{\left(g - \sum \frac{\rho_i}{z - z_i}\right)}_{w} = 0.$$

Theorem

If $d \geqq g + 1$ we can solve $(*)$.

Curvature enters in the proof, which will be given later. We begin with some illustrations of different types of analytic arguments.

A global holomorphic function on C is constant:

- 1-form $\varphi = u\,dx + v\,dy$
  $= \mathcal{U}\,dz + V\,d\bar{z}$
- differential of a function $dw = w_x\,dx + w_y\,dy$
  $= w_z\,dz + w_{\bar{z}}\,d\bar{z}$
- exterior derivative of 1-form[2]
  $d\varphi = (\partial_y v - \partial_x u)\,dx \wedge dy$
  $= (V_z - \mathcal{U}_{\bar{z}})\,dz \wedge d\bar{z}$
- Stokes' theorem $\int_{\partial\mathcal{U}} \varphi = \iint_{\mathcal{U}} d\varphi$, if φ is C^∞ in $\mathcal{U}$.

Proof that a global holomorphic function is constant: $\overline{dw} = d\overline{w}$ and w holomorphic $\implies$

$$(i/2) \iint dw \wedge d\bar{w} = (i/2) \iint |w_z|^2\,dz \wedge d\bar{z}$$

where we are using $\mathcal{U} = C$ in the integral.

[2]Use Liebuitz plus $\alpha \wedge \beta = -\beta \wedge \alpha$ for 1-forms α, β.

Note $(i/2)\, dz \wedge d\bar{z} = dx \wedge dy > 0$. But by Stokes' theorem and $\partial C = 0$

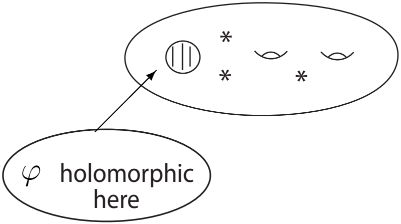
$$\iint dw \wedge d\bar{w} = \iint d(w\, d\bar{w}) = 0$$

$$\implies |w_z|^2 = 0.$$

Residue Theorem

$\sum \mathrm{Res}_p\, \varphi = 0$

Proof: Same as for $\mathbb{P}^1$ using the picture



Where does the "g" come from in $d \geqq g+1$?

Suppose we want w such that

$$w = \frac{a_i}{z - z_i} + \cdots$$

near p_i. For $\varphi \in H^{1,0}(C) \cong \mathbb{C}^g$

$$(**) \qquad \sum_i \mathrm{Res}_{p_i}(w\varphi) = 0$$

gives g equations on $a_1, \ldots, a_d$. Suggests we need $d \geqq g+1$ to have a non-zero solution.

Theorem

$(**)$ *gives necessary and sufficient conditions to have a* $w : C \to \mathbb{P}^1$ *as above.*
We will prove this below.

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Construction of meromorphic differentials

Want φ so that near p_i

$$\varphi = \frac{a_i\, dz}{z - z_i} + \cdots$$

Consider as before

$$\varphi' = \sum_i \rho_i \frac{a_i\, dz}{z - z_i}$$

$$\Phi = \overline{\partial}\varphi' = \sum \overline{\partial}\rho_i \wedge \frac{a_i\, dz}{z - z_i}.$$

If

$$\Phi = \overline{\partial}\eta$$

where η is $C^\infty(1,0)$ form locally looking like

$$u(z, \bar{z})\, dz$$

for $u(z, \bar{z})$ a $\mathbb{C}^\infty$ function, then

$$\varphi = \varphi' - \eta$$

solves the problem.

Now Φ is a global C^∞ 2-form

$$\begin{aligned}\Phi &= H(z,\bar{z})(i/2)\,dz \wedge d\bar{z} \\ &= H(x,y)\,dx \wedge dy\,.\end{aligned}$$

By Stokes' theorem

$$\iint_C \Phi = \lim_{\epsilon \to 0} \iint_{C - \cup \Delta_i(\epsilon)} d\eta = \sum_i a_i .$$

So we need

If $\Phi =$ global C^∞ 2-form on C with $\iint \Phi = 0$, then $\Phi = \overline{\partial}\eta$.

Here is the argument:

- $A^{p,q}$ = global $C^\infty(p,q)$ forms
- $A^{0,0} = A^0$ = global C^∞ functions
- $$\begin{array}{ccc} A^0 \otimes A^{1,1} & \longrightarrow & \mathbb{C} \\ \Cup & & \Cup \\ u \otimes \Phi & \longrightarrow & \iint_C u\Phi \end{array}$$
- $$\begin{array}{ccc} A^{0,1} \otimes A^{1,0} & \longrightarrow & \mathbb{C} \\ \Cup & & \Cup \\ \alpha \otimes \beta & \longrightarrow & \iint \alpha \wedge \beta \end{array}$$

are non-degenerate pairings and

- $A^0 \xrightarrow{\overline{\partial}_0} A^{0,1}$
- $\left\{ \begin{array}{ccc} A^{1,0} & \xrightarrow{\overline{\partial}_1} & A^{1,1} \\ \wr\| & & \wr\| \\ A^{0,1*} & \xrightarrow{\overline{\partial}_0^*} & A^{0*} \end{array} \right.$

are dual. This gives[3]

$$\operatorname{coker} \overline{\partial}_1 \cong \operatorname{coker} \overline{\partial}_0^* \cong (\ker \overline{\partial}_0)^*.$$

We need to make the $A^{p,q}$ into topological vector spaces such that $\overline{\partial}_0$ and $\overline{\partial}_1$ are continuous with closed range; this is the analysis step. Then

$$\ker \overline{\partial}_0 \cong \mathbb{C} \implies \iint \frac{A^{1,1}}{\overline{\partial}_1 A^{1,0}} \xrightarrow{\sim} \mathbb{C}. \qquad \square$$

[3]Dual of a subspace S of a vector space V is isomorphic to the quotient $V^*/S^\perp$ of the dual of V.

- $D = p_1 + \cdots + p_d$, $d \geqq 1$
- $H^{1,0}(C, D) \overset{\text{defn}}{=} \left\{ \begin{matrix} \text{meromorphic differentials} \\ \text{with poles at the } p_i \end{matrix} \right\}$

A similar argument using $\ker\{\overline{\partial}_0 : A^0(-D) \to A^{0,1}(-D)\} = 0$ gives

$$\dim H^{1,0}(C, D) = d + g - 1$$

which is the result about meromorphic differentials with given residues.

This result is a special case of the Riemann-Roch theorem.

The argument contains a special case of the *Kodaira vanishing theorem*: Given

$$\Psi = A^{1,1}(D)$$

where locally $\Psi = w(z, \bar{z})\, dz \wedge d\bar{z} \,/ z - z_i$

$$\implies \Psi = \overline{\partial}\eta$$

where locally $\eta = v(z, \bar{z})\, dz \,/ z - z_i$.

Back to the proof that for $d \geqq g+1$ there exists

$$w : C \to \mathbb{P}^1$$

with $w^{-1}(\infty) = p_1 + \cdots + p_d$. The above argument for

$$\begin{array}{ccc} A^{1,0} & \xrightarrow{\overline{\partial}_1} & A^{1,1} \\ \wr\| & & \wr\| \\ (A^{0,1})^* & \xrightarrow{\overline{\partial}_0^*} & (A^0)^* \end{array}$$

and

$$\ker \overline{\partial}_1 = H^{1,0}(C) \text{ has dimension } g$$

gives

$$(*) \qquad \dim(A^{0,1}/\overline{\partial}_0 A^0) = g.$$

As before consider

$$\eta = \overline{\partial}\left(\sum \rho_i \frac{a_i}{z - z_i}\right) \in A^{0,1}.$$

By $(*)$ there are g conditions on the a_i that there exist $u \in A^0$

$$\eta = \overline{\partial} u.$$

For $d \geqq g + 1$ we have such a u, and then

$$\overline{\partial} \underbrace{\left(\sum \rho_i \frac{a_i}{z - z_i} - u\right)}_{w} = 0$$

$\Longrightarrow$ w gives our desired function. □

Moral

Global existence comes by solving $\overline{\partial}$-equations.

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How does curvature enter?

Will not get in formalism of vector bundles, metrics, connection and curvature matrices. Will use singular metrics and their curvatures in an intuitive way considered as distributions.

- *Metric* on a Riemann surface is locally

$$ds^2 = h\, dz \otimes d\bar{z}, \qquad h > 0.$$

Area form is

$$\Phi = h(i/2)\, dz \wedge d\bar{z}\,.$$

Gauss curvature is

$$(i/2)\overline{\partial}\partial \log h = K\Phi.$$

Gauss-Bonnet is

$$\frac{1}{2\pi}\iint_C K\Phi = \chi(C) = 2 - 2g.$$

Example

On $\mathbb{P}^1$

$$ds^2 = \frac{dz \otimes d\bar{z}}{(1+|z|^2)^2}.$$

Then $K = 1$ and area $= 4\pi$.

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In general curvatures are computed as the (1,1) form

$$\bar{\partial}\partial \log h(z, \bar{z})$$

where $h > 0$ is C^∞. Very useful also is the use of singular metrics, where $h \geqq 0$ and $\log h$ is in L^1 but may have the value $-\infty$.

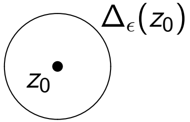
Here the basic formula is (leaving out some constants)

$$\partial\overline{\partial}\log|z-z_0| = \delta_0\, dz\wedge d\bar{z}\,.$$

This means that for any $w \in C_c^\infty(\mathbb{C})$

$$(\sharp) \qquad \iint_{\mathbb{C}} \log|z-z_0|\overline{\partial}\partial w(z,\bar{z}) = w(z_0)$$

Idea:



On $\mathbb{C}\backslash\Delta_\epsilon(z_0)$,

$$\partial\overline{\partial}\log|z-z_0|^2 = \partial\left(\frac{d\bar{z}}{\bar{z}-\bar{z}_0}\right) = d\left(\frac{d\bar{z}}{\bar{z}-\bar{z}_0}\right) = 0.$$

Then LHS of $(\sharp) = \lim_{\epsilon\to 0} \iint_{\mathbb{C}\setminus\Delta_\epsilon(z_0)}(\cdots)$ which using Stokes' theorem and polar coordinates leads to

$$(\sharp) = \lim_{\epsilon\to 0} \int_{\partial\Delta_\epsilon(z_0} w(z,z)\left(\frac{d\bar{z}}{\bar{z}-\bar{z}_0}\right)$$
$$= \lim_{\epsilon\to 0} \oint w\left(z_0 + \epsilon e^{i\theta}\right) d\theta = w(z_0).$$

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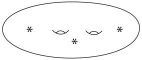
For $D = p_q + \cdots + p_d$  there is a line bundle $\mathcal{O}(D)$ whose sections over an open set $\mathcal{U}$ are the meromorphic functions with poles at the $p_i \in \mathcal{U}$.[4] We define a singular metric by

$$\|w\|^2 = |w(z)|^2.$$

[4]We will not define line bundles, but will only say what their sections over open sets are.

If we change $w(z) \to w(z)u(z)$ where $u(z) \neq 0$ then

$$\|wu\|^2 = |w(z)|^2 |u(z)|^2.$$

Using $\overline{\partial}\partial \log |u(z)|^2 = 0$ the curvature is well defined and is equal to

$$(i/2)\overline{\partial}\partial \log \|w\|^2 = \sum_{p_i \in \mathcal{U}} \delta_{p_i}\big((i/2) dz \wedge d\bar{z}\big)$$

Conclusion: As d increases the curvature in $\mathcal{O}(D)$ becomes more positive and $\mathcal{O}(D)$ has more global sections, these being global meromorphic functions on C with poles on D. A central theme in complex algebraic geometry is

> *positivity of curvature* $\implies$ *existence of global holomorphic objects.*

Note: The Gauss-Bonnet for $\mathcal{O}(D)$ is (up to constants)

$$\iint_C (\text{curvature of } \mathcal{O}(D)) = \deg D = d.$$

The Abel-Jacobi map revisited

- Recall the Abel-Jacobi map

$$\mathrm{AJ} : C \to J(C)$$

defined by

$$\mathrm{AJ}(p) = \begin{pmatrix} \int_{p_0}^{p} \omega_1 \\ \vdots \\ \int_{p_0}^{p} \omega_g \end{pmatrix} \in \mathbb{C}^g / \Lambda$$

where $\omega_1, \ldots, \omega_g$ are a basis for $H^{1,0}(C)$. This is extended to

$$\mathrm{AJ} : C^{(d)} \to J(C)$$

by setting for $D = p_1 + \cdots + p_d$

$$\mathrm{AJ}(D) = \sum_i \mathrm{AJ}(p_i) \qquad \textit{(Abelian sum)}$$

Claim

$$
\begin{array}{ccc}
H^1(C,\mathbb{C}) & \cong H^{1,0}(C) \oplus & \overline{H^{1,0}(C)} \\
\| & & \| \\
\mathrm{Hom}(H_1(C,\mathbb{Z}),\mathbb{C}) & & H^{0,1}(C)
\end{array}
$$

Sketch of proof:

- from topology we know cup-product in cohomology is dual to intersection product in homology
- intersection matrix of standard basis $\{\delta_i, \gamma_j\}$ of $H_1(C,\mathbb{Z})$ is

$$
\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} = Q
$$

- cup product in cohomology is

$$
\alpha \cup \beta = \int_C \alpha \wedge \beta
$$

- using $\omega_i \wedge \omega_j = 0 = \bar{\omega}_i \wedge \bar{\omega}_j$ the cup product matrix for ω_i is

$$\begin{pmatrix} 0 & H \\ -{}^tH & 0 \end{pmatrix}, \quad H_{i,\bar{j}} = \int_C \omega_i \wedge \overline{\omega}_j = -\int_C \overline{\omega}_j \wedge \omega_i$$

- $(i/2)H$ is a *positive definite* Hermitian matrix: For $\omega = h(z)\,dz$

$$(i/2)\int_C \omega \wedge \overline{\omega} = \int_C |h(z)|^2 (i/2)\,dz \wedge d\bar{z}$$

- recall

$$\Omega = \left\| \int_{\delta_i} \omega_k, \int_{\gamma_j} \omega_k \right\| = (A, B)$$

then duality between cup product and intersection pairing gives

$$i\,{}^t\overline{\Omega} Q \Omega = \begin{pmatrix} 0 & H \\ -{}^t\overline{H} & 0 \end{pmatrix}$$

$$\|$$

$$-{}^t\overline{B}A + {}^t\overline{A}B$$

$\implies$ A non-singular

- then may choose basis ω_i to have $A = I$ which gives *normalized period matrix* for $H^{1,0}(C)$

$$\begin{cases} \Omega = (I, Z) \\ Z = {}^tZ, \ \operatorname{Im} Z > 0 \end{cases}$$

- period matrix for $H^{1,0}(C) \oplus \overline{H^{1,0}(C)}$ is

$$\begin{pmatrix} I & Z \\ I & \overline{Z} \end{pmatrix} = \text{non-singular matrix} \qquad \square$$

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- *Riemann theta function:* For $w \in \mathbb{C}^g$

$$\theta(w, Z) = \sum_{m \in \mathbb{Z}^g} \exp\left(2\pi i \left(\frac{1}{2}{}^t mZm + {}^t mw\right)\right)$$

- entire analytic function on $\mathbb{C}^g$[5]
- $\theta(w + \underbrace{a + Zb}_{\substack{\text{general vector in } \Lambda \\ \text{where } a,b \in \mathbb{Z}^g}}) = e^{2\pi i(-{}^t bZ)}\theta(w, Z)$

- $\theta(w, Z) = 0$ defines a hypersurface $\Theta \subset J(C)$
- *Riemann theorem:* $\Theta = \mathrm{AJ}(C^{(g-1)})$.

$\rightsquigarrow$ - *Torelli theorem:* Can reconstruct C from its period matrix.

For $g = 2$, $\mathrm{AJ}(C) = \Theta$.

This is the beginning of how Hodge theory may be used to study the geometry of algebraic curves

——<∘>——

[5] Sum is like $\sum_{m \in \mathbb{Z}^g} e^{-\|m\|^2}$.

Appendix: Notations from complex function theory

- $z = x + iy$ will denote points in open sets in $\mathbb{C}$
- differentials (things you integrate)

$$\begin{cases} dz = dx + i\,dy \\ d\bar{z} = dx - i\,dy \end{cases}$$

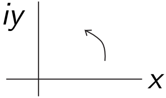
then

$$\begin{cases} dx = \frac{1}{2}(dz + d\bar{z}) \\ dy = \frac{i}{2}(d\bar{z} - dz) \end{cases}$$

- multiplication rule is $\alpha \wedge \beta = -\beta \wedge \alpha$ for 1-forms α, β
- area form is

$$dx \wedge dy = (i/2)\, dz \wedge d\bar{z}$$

- complex plane is oriented



$$, \ (i/2)\, dz \wedge d\bar{z} > 0$$

- $\begin{cases} \partial_z = \frac{1}{2}(\partial_x - i\partial_y) \\ \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) \end{cases}$

$\Longrightarrow \quad \partial_z \partial_{\bar{z}} = \frac{1}{4}(\partial_x^2 + \partial_y^2) = -\Delta$ where $\Delta =$ Laplacian

- $$dw(x, y) = (\partial_x w)\, dx + (\partial_y w)\, dy$$
  $$\|$$
  $$dw(z, \bar{z}) = \underbrace{(\partial_z w)\, dz}_{} + \underbrace{(\partial_{\bar{z}} w)\, d\bar{z}}_{}$$
  $$= \quad \partial w \quad + \quad \bar{\partial} w$$

$\Longrightarrow \quad w$ *holomorphic* $\iff \partial_{\bar{z}} w = 0$

$\iff \bar{\partial}\, w = 0$

Cauchy-Riemann equations

- meromorphic $w(z)$ has Laurent series around $z = 0$

$$w(z) = \sum_{i=-n}^{\infty} a_i z^i$$

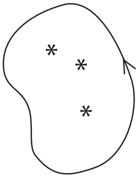
- meromorphic differential

$$\varphi = w(z)\, dz$$

has *residue*

$$\operatorname{Res} \varphi = a_{-1} = \frac{1}{2\pi i} \oint \varphi$$

- *residue theorem*

$\mathfrak{U} \to$ 

$$\boxed{\frac{1}{2\pi i} \int_{\partial \mathfrak{U}} \varphi = \sum_{p \in \mathfrak{U}} \operatorname{Res}_p \varphi}$$