III. Topology and Hodge theory

- These two topics are closely intertwined and constitute a major aspect of complex algebraic geometry, beginning in the later part of the 19\textsuperscript{th} century (Picard, Poincaré, \ldots) into the 1\textsuperscript{st} half of the 20\textsuperscript{th} century (Lefschetz, Hodge, \ldots) and continuing through today.

- In fact questions about integrals on algebraic surfaces (which are real 4-manifolds) were instrumental in the beginnings of topology — one knew (Darboux, Picard, Poincaré, E. Cartan, \ldots) what differential forms

\[
\varphi = a \, dx + b \, dy + c \, dz \\
\psi = A \, dx \wedge dy + B \, dx \wedge dz + C \, dy \wedge dz \\
\eta = D \, dx \wedge dy \wedge dz
\]
were, and Stokes’ theorem

$$\int_{\partial U} d\omega = \int_U \omega$$

shows then when $d\omega = 0$ that $\int_{\Gamma} \omega$ was not only invariant under deformation (homotopy) of $\Gamma$ but also under homology.\(^1\) This led to the notion of *periods*

$$\int_{\Gamma} \omega, \quad d\omega = 0 \text{ and } \Gamma \in H_p(X, \mathbb{Z}).$$

\(^1\)The exterior derivative $d$ is uniquely determined (i) $df = f_x \, dx + f_y \, dy + f_z \, dz$ for a function $f$, (ii) $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$ and (iii) $dx \wedge dy = -dy \wedge dx$ etc.
In the complex case when $X$ has local holomorphic coordinates $z = (z_1, \ldots, z_n)$

$$\omega = \sum_{I,J} f_{I\bar{J}} \, dz^I \wedge d\bar{z}^J$$

where $I = (i_1, \ldots, i_p)$, $dz^J = dz^{i_1} \wedge \cdots \wedge dz^{i_p}$ etc. and as we saw for algebraic curves the periods reflect the complex structure — this is the start of Hodge theory.
Outline for the remainder of this lecture

- Introductory discussion of what an algebraic variety is
- Statements of the Lefschetz theorems
- How they arose historically from the study of algebraic functions of two variables (Picard-Lefschetz or PL theory)
- Origin of the Hodge conjecture (HC)

Complex projective space \( \mathbb{P}^N \)

- lines through origin in \( \mathbb{C}^{N+1} \)
- \( \mathbb{P}^N = \mathbb{C}^N \cup \mathbb{P}^{N-1} \) (\( \mathbb{P}^1 = \mathbb{C} \cup \{ \infty \} \))
- homogeneous coordinates \([z] = [z_0, \ldots, z_N]\)

- \( \mathbb{P}^1 = \text{Riemann sphere} \)
- \( \mathbb{P}^2 = \mathbb{C}^2 \cup \{ \text{lines through the origin} \} \) where \([z] \leftrightarrow \text{line with slope } z_2/z_1\)
- \( \mathbb{P}^N = \text{compact complex manifold} \)
Proof \[ \mathcal{U}_i = \{ [z] : z_i \neq 0 \} \ni [z] \]

\[ \downarrow \quad \downarrow \]

\[ \mathbb{C}^N \ni (z_0/z_i, \ldots \hat{i} \ldots, z_N/z_i) \]

- **Algebraic variety** \( X \subset \mathbb{P}^N \) given by \( F_1(z) = \cdots F_m(z) = 0 \)
  where \( F_\alpha(z) = \) homogeneous polynomial.

- Note that \( \dim_{\mathbb{R}} X = 2 \dim_{\mathbb{C}} X \) and \( X \) is oriented.

**Example**

\( C \) defined by \( f(x, y) = 0 \) in \( \mathbb{C}^2 \). Set

\[ x = z_1/z_0, \quad y = z_2/z_0 \]

and clear denominators to get

\[ \overline{C} = \{ F(z) = 0 \} \subset \mathbb{P}^2 \]
where \( \overline{C} = \left\{ \text{our old } C \subset \mathbb{C}^2 \right\} \cup \left\{ \text{points at } \infty \right\} \).

- suppose \( X^n = \) smooth algebraic variety and \( Y = \mathbb{P}^{N-1} \cap Y \) is a general hyperplane section.
hyperplane section

general

not general

quadric surface; real picture
Note: Equation of the quadric in $\mathbb{C}^3$ is $x^2 + y^2 = z^2 + 1$; equation in $\mathbb{P}^3$ is $z_1^2 + z_2^2 = z_3^2 + z_0^2$; over $\mathbb{C}$ this is equivalent to $z'_1 z'_2 = z'_3 z'_0$ where $z'_1 = z_1 + iz_2$, $z'_2 = z_1 - iz_2$ etc.
Lefschetz theorem I

- \( b_{2p+1}(X) \equiv 0 \) (odd Betti numbers are even)
- \( b_{2p}(X) \geq 1 \) (even Betti numbers are positive).

In the second, if \( \dim \mathbb{C} X = n \) and \( H \in H_{2n-2}(Y, \mathbb{Z}) \) is the class of the cycle given by \( Y \) then (non-trivially)

\[
H \cap \cdots \cap H \neq 0 \text{ in } H_{2p}(X, \mathbb{Z})
\]

Lefschetz theorem II

\( H_p(Y, \mathbb{Z}) \to H_p(X, \mathbb{Z}) \) is

\[
\begin{cases}
\text{isomorphism for } p \leq n - 2 \\
\text{onto for } p = n - 1
\end{cases}
\]

Corollary

Y is connected if \( \dim \mathbb{C} X \geq 2 \)
Exercise: \( f(x, y) = \) irreducible polynomial and
\{ \{ f(x, y) = 0 \} = C \subset \mathbb{C}^2 \). Show that \( C \) is connected.

Geometric idea to study topology of an algebraic variety (idea is one of the most basic in algebraic geometry) — use induction by dimension.

Example

For \( y^2 = p(x) \) where \( p(x) = \prod_{i=1}^{2g+2} (x - a_i) \)

- first take out the two points over \( x = \infty \)
- next use the picture of the complex \( x \)-plane

![Diagram of complex x-plane with points a1, a2, an and 0]
retract the slit \(x\)-plane and the part of \(C\) lying over it onto the part lying over the segments

\[
\begin{align*}
\prod_i T_i &= 1d \\
\Delta_i &= \text{lying over} \quad 0 \rightarrow a_i
\end{align*}
\]

on as we turn around the branch point the two points interchange (local monodromy \(T_i\) around \(a_i\))

\[
\begin{cases}
\text{1-dimensional complex}
\end{cases}
\]
C retracts onto the real 1-dimensional complex given by attaching the $2g + 2$ 1-cells $\Delta_i$ to the two points $\pm$ lying over 0.

$\Delta_i$ generate the relative homology group

$$H_1(C, \{+, -\}; \mathbb{Z}) \sim H_1(C, \mathbb{Z}) \cong \mathbb{Z}^{2g}$$

This case is too simple to suggest the general pattern. The next dimension up is due to Picard (1880–2000)

**Example**

$X$ is the algebraic surface

$$z^2 = f(x, y)$$

where $C = \{f(x, y) = 0\}$ is a non-singular plane curve. For a general $y$ we let

$$X_y = \text{curve } z^2 = f(x, y), \quad y \text{ fixed}$$
The picture is

\[ X_y \text{ is the algebraic curve of the type we have been considering; it is 2:1 covering of the line } y = \text{ constant branched at the points of } C \cap \{ y = \text{ constant} \} \]

- smooth for general \( y \)
- singular when the line \( y = \text{ constant} \) becomes tangent to \( C \)
the picture of $X_y$ is

where the branch points and slits will vary with $y$

at a point of tangency two branch points come together and interchange.
$\gamma \rightarrow \gamma + \delta$

Picard-Lefschetz formula

$(PL)$

$\gamma \rightarrow \gamma + \delta$
How to show PL? The original argument was analytic and in outline went as follows:

- locally analytically change coordinates so that the picture is a neighborhood of the origin of the curves

\[ C_t = \{ u^2 + v^2 = t \} \]

in \( \mathbb{C}^3 \) with coordinates \((u, v, t)\)

- the local picture is

![Diagram]

\[ + \gamma \]
set $t = \sigma^2$ and consider the integrals

$$I_t(\delta) = \int_\delta \frac{du}{\sqrt{t - u^2}} = \int_\delta \frac{du}{\sqrt{\sigma^2 - u^2}} = \int_\delta \frac{du}{\sigma \sqrt{1 - (u/\sigma)^2}}$$

$$I_t(\gamma) = \int_\gamma \frac{du}{\sqrt{t - u^2}} = \int_\gamma \frac{du}{\sqrt{\sigma^2 - u^2}} = \int_\gamma \frac{du}{\sigma \sqrt{1 - (u/\sigma)^2}}$$

the curves $C_t$ are parametrized by

$$z \to (\sigma \sin z, \sigma \cos z),$$

and a calculation gives

$$\begin{cases} I_t(\delta) = 2\pi \\ I_t(\gamma) = i \log t \end{cases}$$
Conclusion

\[
\begin{align*}
I_{e^{2\pi i t}}(\delta) &= I_t(\delta) \\
I_{e^{2\pi i t}}(\gamma) &= I_t(\gamma) + I_t(\delta)
\end{align*}
\]

\[\implies T(\gamma) = \gamma + \delta.\]
Topological pictures

global

local

$\delta \to 0$
few pictures worth 1,000 (10,000?) words

heuristic analytic reasoning suggests what the answer should be — then know what to prove.
$X^* = X \setminus X_\infty$

topological picture of $X^*$

along $\overline{y_0y_i}$ we have the locus of the vanishing cycle $\delta_i = \Delta_i$

$\implies X^*$ obtained from $X_0$ by attaching 2-cells $\Delta_i$
In general

\[ X^* \text{ obtained from } X_0 \text{ by attaching } \]
\[ n = \frac{1}{2} (\dim_{\mathbb{R}} X) \text{ cells} \]

\[ \implies \text{Lefschetz theorems I, II} \]

**Single and double integrals**

Returning to \( X \) given by

\[ z^2 = f(x, y) \]

there are single integrals (1-forms)

\[ \psi = \frac{p(x, y)}{z} \, dx + \frac{q(x, y)}{z} \, dy \]

and double integrals (2-forms)

\[ \varphi = \frac{r(x, y)}{z} \, dx \wedge dy \]
The story of the $\psi$’s is very interesting but we will only have time to make a few observations. For one such we note that

$\int \psi < \infty \implies d\psi = 0.$

**Proof:**

\[
d\psi = d \left( \frac{p(x, y)}{z} \right) \wedge dx + d \left( \frac{q(x, y)}{z} \right) \wedge dy
\]

\[
= \frac{r(x, y)}{z} \ dx \wedge dy
\]

$\implies \frac{1}{4} (d\psi \wedge \overline{d\psi}) = \left| \frac{r(x, y)}{z} \right|^2 \left( \frac{i}{2} \right) dx \wedge d\overline{x} \wedge \left( \frac{i}{2} \right) dy \wedge d\overline{y}$

$= \text{volume form on } X$

\[0 < \int_X d\psi \wedge \overline{d\psi} = \int_X d(\psi \wedge \overline{d\psi}) = 0 \implies d\psi = 0.\]
The space of single integrals is denoted by $H^{1,0}(X)$ and its dimension $h^{1,0}(X)$ is called the *irregularity* — reason for the name is that in the early days “most” surfaces seemed to be *regular*, i.e., to have $h^{1,0}(X) = 0$.

**Example**

For $z^2 = f(x, y)$ to be irregular the curve $C$ cannot be smooth, or even have generic singularities, those being where

$$
\begin{align*}
 f_x(x, y) = f_y(x, y) &= 0 \\
 \det \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{xx} \end{vmatrix}(x, y) &\neq 0
\end{align*}
$$

Similarly for a hypersurface

$$
F(z_0, z_1, z_2, z_3) = 0
$$

in $\mathbb{P}^3$ it is not easy to write down on $F$ where $X$ is irregular.
Suppose now \( \varphi \) is a regular 2-form; i.e.,
\[
\int_\sigma \varphi < \infty
\]
for any 2-chain \( \sigma \). We set
\[
H^{2,0}(X) = \left\{ \begin{array}{l}
\text{space of} \\
\text{regular 2-forms}
\end{array} \right\}.
\]
The *periods* of \( \psi \) are the
\[
\int_\Gamma \psi, \quad \Gamma \in H_2(X, \mathbb{Z}).
\]
Among the \( \Gamma \)'s are the fundamental classes of algebraic curves \( C \subset X \); i.e., the images of
\[
H_2(C, \mathbb{Z}) \to H_2(X, \mathbb{Z}).
\]
We will discuss these further below.

- By restriction

\[
\psi \rightarrow \psi_y = \frac{p(x, y) \, dx}{z}
\]

we will generally have \( \psi_y \neq 0 \) which gives

\[
H^{1,0}(X) \hookrightarrow H^{1,0}(X_y).
\]

This suggests that we have

\[
H^1(X, \mathbb{C}) \hookrightarrow H^1(X_y, \mathbb{C}),
\]

which is true and is what originally suggested the first non-easy case of Lefschetz II — again analysis and topology went hand in hand.
Another example of the use of analysis to suggest topology:

For a vanishing cycle

\[ \Delta_i = \ \ \ \ ]

traced out by \( \delta_y \in H_1(X_y, \mathbb{Z}) \) along the path from 0 to \( a_i \)

we have

\[ \int_{\delta_0} \psi = \int_{\delta_{a_i}} \psi = 0, \quad \psi \in H^{1,0}(X). \]
This led to Picard’s argument that

$$\ker\{H_1(X_0, \mathbb{Z}) \to H_1(X, \mathbb{Z})\} = \left\{ \begin{array}{l}
\text{span of the} \\
\text{space of vanishing cycles.}
\end{array} \right\}$$

Returning to the discussion of

- Among the classes in $H_2(X, \mathbb{Z})$ are those given by the fundamental classes of the algebraic curves $C$ contained in $X$. 

Example:

\[ \begin{array}{c}
\{ \text{two families on lines on a quadric surface} \\
\quad z_0 z_1 = z_2 z_3 \end{array} \] \]

\[ \leadsto H_2(X, \mathbb{Z}) \cong \mathbb{Z}[L_1] \oplus \mathbb{Z}[L_2] \]

- In general, \( C \) is a component of

\[ \begin{cases}
 z^2 = f(x, y) \\
 g(x, y, z) = 0
\end{cases} \]

(may take \( g(x, y, z) = g_0(x, y) + g_1(x, y)z \))

\[ ^2 \text{The lines are } z_0 = z_2 = 0, [z_1, z_3] \in \mathbb{P}^1 \text{ arbitrary and } z_1 = z_3 = 0, [z_0, z_2] \text{ arbitrary.} \]
On $X$

\[ 0 = dg = g_x \, dx + g_y \, dy + g_z \, dz \]

which using $dz = \left( \frac{1}{2z} \right) (f_x \, dx + f_y \, dy)$ gives a relation

\[ a \, dx + b \, dy \bigg|_C = 0 \]

\[ \Rightarrow \quad \psi \bigg|_C = 0 \]

\[ \Rightarrow \quad \int_{[C]} \psi = 0. \]
Conclusion: The periods of $H^{2,0}(X)$ on the homology classes of algebraic curves are equal to zero.

- The converse statement is the famous Lefschetz (1,1) theorem.
- The converse to the analogous statement for arbitrary $X$ is the Hodge conjecture.
- In terms of differential forms of degree 2 on $X$ there are three types:
  - $p(x,y) \frac{dx \wedge dy}{z} \leftrightarrow H^{2,0}(X)$
  - conjugates of these $\leftrightarrow \overline{H^{2,0}(X)} = H^{0,2}(X)$
  - those that have a $dx \wedge d\bar{x}, dx \wedge d\bar{y}, d\bar{x} \wedge dy, dy \wedge d\bar{y}$ which are said to be of type (1,1) and contribute $H^{1,1}(X)$ to $H^2(X, \mathbb{C})$; it is these that are Poincaré dual to the homology classes carried by the algebraic curves in $X$. 
Further topics

- These involve the *multiplicative structure* on cohomology: For $X$ of dimension $n$ and $H \in H^2(X)$ the class of a hyperplane section

\[(*) \quad L^k : H^{n-k}(X) \to H^{n+k}(X).\]

**Hard Lefschetz theorem:** $(*)$ *is an isomorphism*

Lefschetz stated the result but his proof was incomplete. Hodge developed Hodge theory to prove $(*)$.

- Define operators $L, H, \Lambda$ on $H^*(X)$ by
  - $L$ as above
  - $H = (d - n) \text{Id}$ on $H^d(X)$
Then the commutator

\[ [H, L] = 2L. \]

There is a unique \( \mathfrak{sl}_2 = \{ L, H, \Lambda \} \) with

\[
\begin{cases}
[L, \Lambda] = H \\
[H, \Lambda] = -2\Lambda.
\end{cases}
\]

Decomposing \( H^*(X) \) into irreducible \( \mathfrak{sl}_2 \)-modules gives the Lefschetz decomposition of cohomology into primitive subspaces — every class is a linear combination of powers of \( L \) applied to primitive classes

\[
\begin{cases}
L^k \cdot \eta \\
\Lambda \eta = 0.
\end{cases}
\]
Any irreducible $\mathfrak{sl}_2$-module is isomorphic to

- $V = \text{span}\{x^n, x^{n-1}y, \ldots, xy^{n-1}, y^n\}$
- $L = \partial_x$, $\Lambda = \partial_y$
- primitive part is generated by $x^n$.

**Example:** $X = $ algebraic surface

$$H^1(X) \xrightarrow{\sim} H^3(X)$$

and

$$H^0(X) \overset{L}{\rightarrow} H^2(X) \overset{L}{\rightarrow} H^4(X)$$

has

- $H^2(X)_{\text{prim}} = \ker\{H^2(X) \xrightarrow{L} H^4(X)\}$
- $H^2(X) = LH^0(X) \oplus H^2(X)_{\text{prim}}$
Finally, you may ask: OK, we know a lot about the homology of $X$ — what about its homotopy?

**Theorem**

*The rational homotopy type of $X$ is uniquely determined by $H^\ast(X)$.***

Thus the

- $\pi_i(X) \otimes \mathbb{Q}$
- Massey triple products $/\mathbb{Q}$, etc. are all equal to zero

$\implies$ Very strong homotopy-theoretic conditions that $X$ be topologically a smooth algebraic variety.
Appendix: Monodromy

- $C_0 = \text{smooth algebraic curve over the origin}$

- fundamental group $\pi_1 = \pi_1(\mathbb{C}\setminus\{\text{slits}\})$ acts on $H_1(C_0, \mathbb{Z})$

- action of $\pi_1$ is generated by PL transformation

  $$T_i : \gamma \rightarrow \gamma + (\gamma, \delta_i)\delta_i$$

- $\prod T_i = \text{identity}$

- action of $\pi_1$ preserves the intersection form

  $$Q : H_1(C_0, \mathbb{Z}) \otimes H_1(C_0, \mathbb{Z}) \rightarrow \mathbb{Z}$$

- Invariant cycles

  $$H_1(C_0, \mathbb{Q})^{\text{inv}} = \text{span}\{\gamma : (\gamma, \delta_i) = 0 \text{ for all } i\}$$
Vanishing cycles

\[ H_1(C, \mathbb{Q})^{\text{van}} = \text{span}\{\delta_i\} \]

If we know that

\[(*) \quad H_1(C_0, \mathbb{Q})^{\text{van}} \cap H_1(C_0, \mathbb{Q})^{\text{inv}} = (0)\]

then

\[ Q = \begin{pmatrix} * & * = 0 \\ 0 & * \end{pmatrix} \]

and the monodromy representation is semi-simple

Lefschetz stated \((*)\) but his proof was incomplete — in fact

\((*)\) is true, but its proof requires analysis

The analysis was provided by Hodge.
It is a general fact proved by Deligne in the geometric case and by Schmid in general that *general monodromy representations are always semi-simple*.

The proofs require Hodge theory and are among the most basic properties of the topology of algebraic varieties.

The reason Lefschetz wanted to have the result is that

\[(\ast) \iff \text{Hard Lefschetz}\]

Lefschetz proof of this assertion was correct.
Hodge structure of weight \( n \)

\( (V, F^p) \)

\( V = \mathbb{Q} \)-vector space (frequently \( \mathbb{Q}_2 \))

\( F^p \) = decreasing filtration \( \forall c, 0 \leq p \leq n \)

\( F^p \cap F^{n-p+2} = V_c \)

\( V_{p, q} = F^p \cap F^q = \tilde{V}^{p, q} \)

\( V_c = \bigoplus V_{p, q} \)

\( F^p = \bigoplus_{p' \geq p} V_{p', q} \)

\( \text{Ex} \ H^n(X, \mathbb{Q}) \)

Polarized Hodge structure of weight \( n \)

\( (V, Q, F^p) \)

\( \{ Q(F^p, F^{n-p+2}) = 0 \quad \text{(I)} \)  

\( \text{deg} Q(V_{p, q}, \overline{V}_{p, q}) > 0 \quad \text{(II)} \)

\( \text{Ex} \ H^n(X, \mathbb{Q})_{\text{prim}} \)
Mixed Hodge structure

\[(V, W, F^p)\]

\[W = \text{maximal filtration} / 0\]

\[G^m W \cong W_m V / W_{m-1} V\]

\[F^p \text{ induces on } G^m W \text{ a HS of weight } m\]

Ex 2 complete, \(V = H^m(\mathbb{X}, \mathcal{O})\)
and \(0 \leq m \leq n\)

Ex 3 smooth and affine, \(V = H^m(\mathbb{X}, \mathcal{O})\)
and \(n \leq m \leq 2n\)

Derived domain

\[D = \{ F' : (V, \mathcal{O}, F') = \text{PHS} \}\]

\[D = G_{\mathbb{R}} / H, \quad G_{\mathbb{R}} = G(\mathbb{R}) \text{ where}\]

\[G = \text{Aut} (V, \mathcal{O}) \text{ and}\]

\[H = \text{unipotent of } F_0 \subset D\]

Ex \(n = 1\)

\[V_c = V_{c,0} \oplus \bar{V}_{c,0}\]

\[Q(V_{c,0}, V_{c,0}) = 0, \quad Q(V_{c,0}, \bar{V}_{c,0}) > 0\]
\[ 
\rightarrow D = S^g / U(g) = \mathcal{D}^g 
\]

Ex: \( n=2 \)
\[ V_c = V^{1,0} \oplus V^{2,0} \oplus \overline{V^{2,0}} \]
\[ \dim a \quad \dim b \]

\[ \rightarrow D \cong SO(2a, b) / U(a) \times SO(b) \]

Compact dual
\[ \tilde{D} = \{ \phi : G(F, E^{n-k+2}) = 0 \} \]
\[ \tilde{D} = G_c \slash D, \quad p = \text{parabolic} \]

\[ \rightarrow D = \text{open} \quad G_{\mathbb{R}} \text{-orbit in} \tilde{D} \]

\( G_{\mathbb{R}} \)-orbit structure of \( \tilde{D} \) is a rich subject

Period mapping
\[ \chi \rightarrow B, \quad \pi^{-1}(b) = \mathcal{R}_{b} \text{ smooth} \]

monodromy representation
\[ \xi : \pi_1(B) \rightarrow \text{Aut}(H^0(X_0, G)_{\text{prin}}, G) \]

\[ \rightarrow \xi : B \rightarrow \Gamma \backslash D \]

where \( \xi(\pi_1(B)) < \Gamma < G \)
**Limiting Mixed Hodge Structure (LMHS)**

- Nilpotent $N$ gives unique $W(N)$
  - $N : W_m \rightarrow W_{m-1}(N)$
  - $N^k : G_{n+k} V \rightarrow G_{n+k} W(N)$
- There exists $\mathfrak{m}_N = \{ N, H, N^* \}$ such that $N = \mathfrak{m}_N$

LMHS = $\{ V, W(N), F_{\mathfrak{m}_N} \}$

**Ex:** $E : \Delta^* \rightarrow K^2 \setminus D$ gives LMHS (Schmid)
Local monodromy cycle theorem

\[ f_t: \mathbb{F} \to \mathbb{F}_0 \]

\[ \mathbb{F} \to \mathbb{C} \]

\[ \Rightarrow f^* H^\text{\textit{\textalpha}}(\mathbb{F}_0) = \text{ker}N \]

Several variable case

\[ \Delta^* \times \Delta^2 \to \Delta \setminus \Delta^2 \]

- \( W(N_\lambda) \) independent of \( \lambda \) when
  \[ N_\lambda = \sum \lambda_i N_i, \quad \lambda_i > 0 \]

- Relative weight filtration property

- Asymptotics of degenerating PHS's
  (Cattani-Kaplan-Schnid)