IV. What is the Hodge conjecture, and why hasn't it been proved?

Short answer

- the HC proposes necessary and sufficient conditions that a homology class be represented by an *algebraic cycle* (a linear combination of the fundamental classes of algebraic subvarieties)
- in codimension 1 the result is the Lefschetz (1,1) theorem — for codimension $\geqq 2$ there are new Hodge-theoretic invariants of algebraic cycles of an *arithmetic character* and these are not understood.

- it is known that the HC has implications for these arithmetic invariants, but it is not understood what, if any, direct implications they have for the HC
- the issue boils down to constructing something under assumptions that have both a geometric and an arithmetic aspect.

There is basically one case of a variant of the HC beyond the codimension 1 case that is understood — this can be analyzed using classical complex analysis plus some arithmetic and will be the main topic of today's lecture

Outline

A. The Hodge conjecture (HC)

B. Relative Chow groups for $(\mathbb{P}^1, \{0, \infty\})$ and $(\mathbb{P}^2, T)$.

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A: The HC

- X = smooth n-dimensional complete algebraic variety (thus it is a compact $2n$-real dimensional manifold)
- $H^r(X, \mathbb{C}) \cong H^r_{\mathrm{DR}}(X)$ where the RHS is

$$H^r_{\mathrm{DR}}(X) = \left\{ \frac{Z^r(X)}{dA^{r-1}(X)} \right\} = \frac{\left\{ \begin{array}{c} \text{closed } r\text{-forms; i.e.,} \\ \text{those } \omega \text{ with } d\omega = 0 \end{array} \right\}}{\left\{ \begin{array}{c} \text{exact } r\text{-forms} \\ \omega = d\psi \end{array} \right\}}$$

- for X = complex manifold with local holomorphic coordinates $z_1, \dots, z_r$
  - $A^r(X) = \bigoplus_{p+q=r} A^{p,q}(X)$
  - $$A^{p,q}(X) = \Big\{ \Psi = \sum_{\substack{|I|=p \\ |J|=q}} \Psi_{I\bar{J}}\, dz^I \wedge d\bar{z}^J \Big\} = \overline{A^{q,p}(X)}$$
    (decomposition into (p,q) types)
  - for X a smooth complete algebraic variety this (p,q) decomposition descends to cohomology

    $$H^r(X,\mathbb{C}) \cong \underbrace{\bigoplus_{p+q=r} H^{p,q}(X), \qquad H^{p,q}(X) = \overline{H^{q,p}(X)}}_{\text{Hodge decomposition on cohomology}}$$

    Thus $H^r(X,\mathbb{C})$ has a *Hodge structure of weight r*

- For X any algebraic variety $H^r(X)$ has a *mixed Hodge structure* where

$$X \begin{cases} \text{complete} \implies \text{weights are } 0 \leqq w \leqq r \\ \text{smooth but open} \implies r \leqq w \leqq 2r \end{cases}$$

- There is also a mixed Hodge structure for the cohomology of relative algebraic varieties; we will implicitly be using this later.
  - $H_{2n-r}(X) \cong H^r(X)$ (Poincaré duality)
  - $Y \subset X$ an $(n-r)$-dimensional subvariety $\rightsquigarrow [Y] \in H_{2(n-r)}(X) \cong H^{2r}(X)$ (recall that $\dim_{\mathbb{R}} Y = 2(n-r)$)
  - $[Y] \in H^{r,r}(X)$ (Y locally given by $z_1 = \cdots = z_r = 0$)
  - *Hodge classes*

$$\mathrm{Hg}^r(X) = H^{2r}(X, \mathbb{Q}) \cap H^{r,r}(X).$$

Example: $X =$ algebraic surface

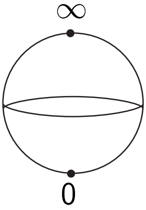
$$H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

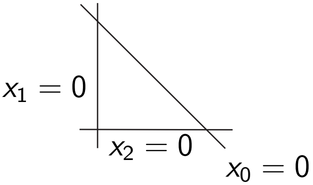
- $H^{2,0}(X) =$ regular 2-forms
- $H^{0,2}(X) = \overline{H^{2,0}(X)}$
- $\left\{ \begin{array}{l} H^{1,1}(X) \text{ is there to represent} \\ \text{the fundamental classes of} \\ \text{the algebraic curves on } X \end{array} \right\}$

- *Hodge conjecture:* $\mathrm{Hg}^r(X)$ is generated by fundamental classes of codimension-r subvarieties on X
- due to Lefschetz when $r = 1$ — essentially no other known cases — there are a few examples — it is non-trivially consistent with known consequences.

Issue: Have to construct something — it is an *existence* result — for $r \geqq 2$ there is an arithmetic aspect and thus far existing methods of complex analysis/PDE/differential geometry fall short.

B: $(\mathbb{P}^1, \{0, \infty\})$ and $(\mathbb{P}^2, T)$





- $[x_0, x_1]$
- $\begin{cases} 0 \leftrightarrow x_1 = 0 \\ \infty \leftrightarrow x_0 = 0 \end{cases}$
- $z = x_1/x_0$

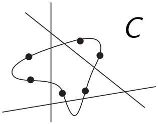
- $[x_0, x_1, x_2]$
- $\begin{cases} x = x_1/x_0 \\ y = x_2/x_0 \end{cases}$
- Line at infinity is $x_0 = 0$, and then $[0, x_1, x_2]$ gives the direction in $\mathbb{C}^2$ to go to that point on the line at infinity.

- 0-cycles are $D = \sum_i n_i p_i$, $n_i \in \mathbb{Z}$ and

$$p_i \in \begin{cases} \mathbb{P}^1 \backslash \{0, \infty\} \\ \mathbb{P}^2 \backslash T \end{cases}$$

- set $D_+ = \sum n_i p_i$, $n_i > 0$ and $D_- = \sum n_i p_i$, $n_i < 0$
- for $(\mathbb{P}^1; \{0, \infty\})$ we want to construct a rational function $w(z)$ such that
  (i) $(w) = D$
  (ii) $w =$ const. on $\{0, \infty\}$ (i.e., $w(0) = w(\infty)$)
- note that if w, w' have $(w) = D$, $(w') = D'$ and w, w' are constant on $\{0, \infty\}$, then $(ww') = D + D'$, $(w/w') = D - D'$ and w/w' is constant on $\{0, \infty\}$

- for $(\mathbb{P}^2, T)$ we want to construct a pair (C, w) where
    - C is an algebraic curve with $C^* = C \backslash C \cap T$ (C may not be irreducible)



- $p_i \in C^*$
- a rational function $w = \left.\frac{p(x,y)}{q(x,y)}\right|_C$ such that
    (i) $(w) = D$
    (ii) $w =$ const. on T

Writing

$$D = D_+ - D_-$$

in both cases we have a rational family $D_t = w^{-1}(t)$ of 0-cycles where $D_0 = D_+$, $D_\infty = D_-$ (this is called a *rational equivalence*, written $D \sim 0$). In the $(\mathbb{P}^2, T)$ case as t varies over $\mathbb{P}^1$ the D_t will lie on a curve C.

- Again if $D \sim 0$, $D' \sim 0$, then $D \pm D' \sim 0$.

The group of 0-cycles D modulo rational equivalence is the *Chow group* $\mathrm{CH}_0(\mathbb{P}^2, T)$.
In this example the curves C we need will not be mysterious; they will be configurations of lines.

Interlude: Recall Abel's theorem:

$$\sum_i \int_{(x_0,y_0)}^{(x_i(t),y_i(t))} \omega = \text{constant}$$

where $\omega = r(x, y(x))\, dx$ is a regular 1-form on the algebraic curve $f(x, y) = 0$ (regular means that $\int \omega < \infty$), and

$$D_t \overset{\text{defn}}{=} \sum_i (x_i(t), y_i(t)) = \{g(x, y, t) \cap f(x, y)\}$$

are the intersection points of C with a family of algebraic curves $g(x, y, t) = 0$ depending *rationally* on a parameter.

- Converse to Abel's theorem:

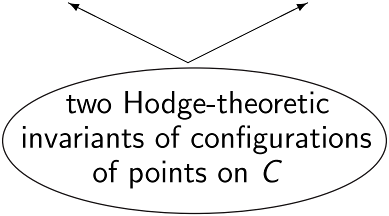
  Given $D = \sum^{d} p_i$*,* $D' = \sum^{d'} p'_i$ *with* $\deg D = \deg D'$ *and* $\mathrm{AJ}(D - D') = 0$ *in* $J(C)$*, there exists a rationally parametrized family* D_t *with* $D = D_0$*,* $D' = D_\infty$*.*

In fact there exists a meromorphic function $w : C \to \mathbb{P}^1$ with $w^{-1}(0) = D$, $w^{-1}(\infty) = D'$. Thus $\mathrm{CH}_0(C) = J(C)$.

In general as noted above the *Chow group* of an algebraic variety is generated by the group of 0-cycles $Z = \sum_i n_i p_i$ modulo the relation $Z \sim Z'$ generated by moving Z to Z' by a rational parameter.

Summarizing the story for algebraic curves we have

$$0 \to J(C) \to CH_0(C) \xrightarrow{\deg} H_0(C, \mathbb{Z}) \to 0$$ [1]



For algebraic surfaces there will be *three* Hodge-theoretic invariants corresponding to integrating 0-forms, 1-forms and 2-forms, and

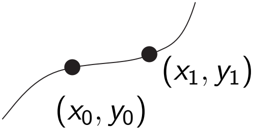
the third one will be arithmetically defined

It is the relation between the integrals of algebraic functions and arithmetic that is a (the?) missing piece.

[1] $\deg D = \int_D 1$

Interlude:

- Suppose $f(x, y) \in \mathbb{Q}[x, y]$ has rational coefficients (or they could be in $k =$ finite extension of $\mathbb{Q}$ such as $\mathbb{Q}(\sqrt{a})$ etc.)
- $\omega = r(x, y(x))\, dx$ where $r(x, y) \in \mathbb{Q}[x, y]$
- $(x_0, y_0) \in C$ is a rational point



- $(x_1, y_1) \in C$ close to (x_0, y_0) another rational point.

Theorem: (*many people including Siegel*). Assume $\int \omega$ is *not* an algebraic function of the upper limit. Then

$$I(x_1, y_1) = \int_{(x_0,y_0)}^{(x_1,y_1)} \omega \textit{ is not an algebraic number.}$$ [2]

- Variant: Only finitely many relations

$$\sum_i a_i I(x_i, y_i) = 0, \qquad a_i \in \mathbb{Q}.$$

- Conjecture: Relations come from geometry.
- This gives a conjecturally deep geometric relation between periods and arithmetic.

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[2] We may view $I(x_1, y_1)$ as a period for the relative curve $(C, \{(x_0, y_0), (x_1, y_1)\})$.

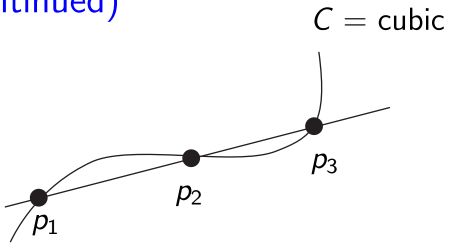
Recall

$$\mathbb{C}/\Lambda \xrightarrow{(p(u),p'(u))} C \subset \mathbb{P}^2.$$

Theorem has the

Corollary: *$p(u)$ algebraic $\implies$ u transcendental.*[3]

Example (continued)

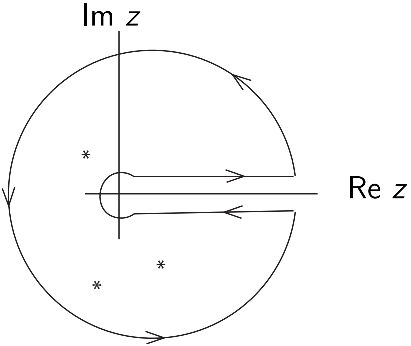


[3] This is the tip of the iceberg of a deep story about the arithmetic properties of periods and the values of transcendental functions that are solutions of algebraic PE's defined $/\overline{\mathbb{Q}}$ ($(p')^2 = p^3 + ap + b$ in this case — Picard-Fuchs equations in general).

Abel: $\sum_{i=1}^{3} \int^{p_i} \omega = 0.$

Chow group of $(\mathbb{P}^1; \{0, \infty\})$

- for $w(z) = \prod (z - z_i)^{n_i}$ write $D = \sum n_i z_i$ and set $\deg D = \sum_i n_i$
- in the picture in the complex plane



$$
\begin{aligned}
0 = \frac{1}{2\pi i}\oint \frac{dw(z)}{w(z)} &= \sum \mathrm{Res}\left(\frac{dw}{w}\right) \\
&= \sum_i n_i
\end{aligned}
$$

- $\implies \mathrm{AJ}_0(D) = \deg D = 0$ ($\#$ zeroes $= \#$ poles)
- for same figure now choose a single-valued branch of $\log z$ and set

$$\psi = \log z \frac{dw(z)}{w(z)}$$

- $0 = \frac{1}{2\pi i}\oint \psi = \sum n_i \log z_i$
  $\implies \mathrm{AJ}_1(D) = \prod_i z_i^{n_i} = 1$
- the mixed Hodge structure for $H^1(\mathbb{P}^1; \{0, \infty\})$ is generated by $\omega = dz/z$, and then in general $\mathrm{AJ}_1(D)$ $= \sum n_i \int_{z_0}^{z_i} \omega \bmod 2\pi i$; thus $\mathrm{AJ}_1(D) = 0 \iff \prod z_i^{n_i} = 1$.

Thus both “deg” and “AJ” have Hodge-theoretic meaning. The above result is expressed by

$$\begin{array}{l} 1 \to \mathbb{C}^* \to \mathrm{CH}_0(\mathbb{P}^1; \{0, \infty\}) \to \mathbb{Z} \to 0 \\ \qquad \| \\ J((\mathbb{P}^1; \{0, \infty\})) \end{array}$$

- the simplest 0-cycles in $\ker(\deg) \cap \ker(\mathrm{AJ}_1)$ are the

$$\begin{aligned} D &= a + b - 1 - ab \\ &= (a-1) + (b-1) - (ab-1) \\ &= D_a + D_b - D_{ab}, \end{aligned}$$

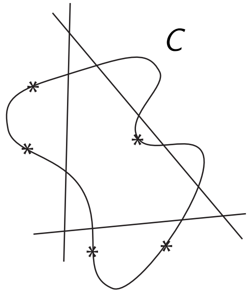
then

$$w(z) = \frac{(z-a)(z-b)}{(z-1)(z-ab)}$$

has $(w) = D$ as above.

Chow group for $(\mathbb{P}^2, T)$

- set $p_i = (x_i, y_i) \in \mathbb{C}^* \times \mathbb{C}^*$



- the particular type of curve C will enter the story later; for now we just consider a rational function $w(x, y) = \frac{p(x,y)}{q(x,y)}$ restricted to any C and with divisor $D = \sum n_i p_i$
- as usual the residue theorem on C for dw/w gives

$$\sum_i n_i = 0$$

- next the residue theorem for $\log x \frac{dw}{w}$ and $\log y \frac{dw}{w}$ gives[4]

$$\prod x_i^{n_i} = 1, \qquad \prod y_i^{n_i} = 1$$

- At this point the issue becomes rather subtle. Set
  - $\mathrm{Div}_0(\mathbb{P}^2, T) =$ 0-cycles of degree 0
  - $$\begin{array}{ccc} \mathrm{Div}_0(\mathbb{P}^2, T) & \xrightarrow{\mathrm{AJ}_1} & \mathbb{C}^* \times \mathbb{C}^* \\ \cup & & \cup \\ D & \longrightarrow & (\prod x_i^{n_i}, \prod y_i^{n_i}) \end{array}$$
- The D_a's above are

$$D_{a,b} = (a, b) - (a, 1) - (1, b) + (1, 1).$$

They generate a subgroup

$$\ker(\mathrm{AJ}_0) \cap \ker(\mathrm{AJ}_1)$$

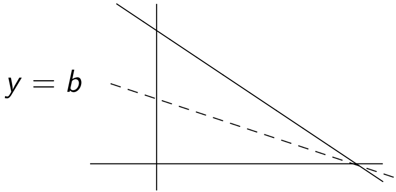
of $\mathrm{Div}_0(\mathbb{P}^2, T)$, where we set $\mathrm{AJ}_0 = \deg$.

[4] Below we will interpret this in terms of the differentials dx/x and dy/y that give the mixed Hodge structure on H^1.

- We consider the rational function

$$\frac{(x-a_1)(x-a_2)}{(x-1)(x-a_1a_2)}$$

on the curve $C = \{y = b\}$



This gives

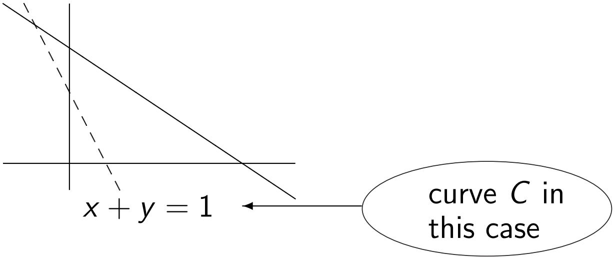
$$D_{a_1,b} + D_{a_2,b} \sim D_{a_1a_2,b}$$
$$D_{a^2,b} \sim D_{a,b} + D_{a,b} \sim D_{a,b^2}$$

Conclusion: *The map*

$$\mathrm{Div}_0(\mathbb{P}^2, T)/_\sim \to \mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{C}^*$$

is well defined.

- It would have been simpler if the story had ended here. But essentially we have only used the lines through the vertices of the triangle T. Consider now



For

$$w = \prod (x - a_i)^{n_i} \Big|_{x+y=1}$$

where $\sum n_i = 0$, $\prod a_i^{n_i} = 1 = \prod (1 - a_i)^{n_i}$ we get

$$\sum_i D_{a_i, 1-a_i} \sim 0.$$

This intertwines x, y in a subtle way.

Definition: $K_2(\mathbb{C}) = \mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{C}^* \big/ \{a \otimes (1-a)\}$ where $a \neq 0, 1, \infty$ (i.e., $a \in \mathbb{C}^* \backslash \{1\}$).

The relations $a \otimes (1-a) \sim 1$ are the *Steinberg relations*.

Theorem: $\mathrm{AJ}_2 : \mathrm{CH}(\mathbb{P}^2, T) \xrightarrow{\sim} K_2(\mathbb{C})$

- Conjecturally AJ_2 can also be defined Hodge-theoretically (see below).

- The group $K_2(\mathbb{C})$ is a subtle *arithmetic* object. Setting $\{a, b\}$ = image of $a \otimes b$ in $K_2(\mathbb{C})$ one has
  - $\{a, 1\} = 1 = \{1, b\}$
  - (∗) $\{a, b\} = 1$ if $a, b \in \overline{\mathbb{Q}}$.

  To prove the first relation and illustrate why the second relation might hold, on $x = y$

$$(ab, ab) - (a, a) - (b, b) + (1, 1) \sim 0$$

$$\begin{aligned} &\implies D_{a,b} + D_{b,a} \sim 0 \,^{5} \\ &\implies \{a, b\} = \{b, a\}^{-1} \\ &\qquad\qquad\quad = \{1/b, a\} \end{aligned}$$

  Then

$$\begin{aligned} \{a, 1\} &= \{a, 1-a\}\{a, 1/1-a\} \\ &= \{a, 1-a\}^{-1} \\ &= 1. \end{aligned}$$

[5] This requires a little calculation.

For $\lambda^n = 1$

$$\begin{aligned} 1 &= \{a, 1\} = \{a, \lambda\}^n \\ \implies & \{a, \lambda\} \text{ is torsion.} \end{aligned}$$

This is a step towards showing $(*)$.

Corollary: *Given $x_i, y_i \in \overline{\mathbb{Q}}$, $n_i \in \mathbb{Z}$ such that $\sum_i n_i = 0$, $\prod_i x_i^{n_i} = \prod_i y_{y_i}^{n_i} = 1$, there exists a curve C, and on C a rational function w such that $(w) = \sum n_i(x_i, y_i)$.*

This is not the case without the assumption $x_i, y_i \in \overline{\mathbb{Q}}$ — we now discuss a Hodge-theoretic construction that proves that for general $D = \sum_i n_i(x_i, y_i)$ where the x_i, y_i are *not* algebraic, we do *not* have $D \sim 0$.

Hodge-theoretic interpretation in terms of periods

- For
$$D = \sum_i n_i p_i = \sum_i n_i(x_i, y_i)$$
we first have that the two classical Hodge-theoretic assumptions
  - $\mathrm{AJ}_0(D) = \deg D = \int_D 1 = \sum_i n_i = 0$ where $1 \in H^0(\Omega^0_{X^*})$
  - $\mathrm{AJ}_1(D) = \left(\int_\gamma \frac{dx}{x}, \int_\gamma \frac{dy}{y}\right) \equiv 0 \left\{ \begin{matrix} \text{mod} \\ \text{periods} \end{matrix} \right\}$ where $\frac{dx}{x}, \frac{dy}{y} \in H^0(\Omega^1_{X^*})$ and $\partial\gamma = D$

  are necessary to have $D \sim 0$, but by the theorem above they are not sufficient unless the $x_i, y_i \in \overline{\mathbb{Q}}$.
- The remaining part of the Hodge theory of $(\mathbb{P}^2, T)$ is given by
$$\omega = \frac{dx}{x} \wedge \frac{dy}{y} \in H^0(\Omega^2_{X^*}).$$

This raises the question: *Is there an "Abel-Jacobi" map involving ω that gives the remaining necessary and sufficient conditions to have $D \sim 0$?*

The answer to this is only conjecturally known. The issue is to construct something that is both geometric and arithmetic (more precisely, to construct something geometric $/\mathbb{C}$ and arithmetic $/\mathbb{Q}$).

Spreads: Given $D = \sum n_i(x_i, y_i)$ as above the x_i, y_i generate a subfield $k \subset \mathbb{C}$. This field has finite transcendence degree; thus

$$k \cong \mathbb{Q}\Big[\underbrace{\alpha_1, \dots, \alpha_n}_{\substack{\text{independent}\\ \text{transcendentals}}} \; ; \; \underbrace{\beta_1, \dots, \beta_\ell}_{\substack{\text{algebraic over}\\ \mathbb{Q}[\alpha_1, \dots, \alpha_n]}}\Big]$$

where $\mathrm{Tr}\deg(k/\mathbb{Q}) = n$.

Using the equations that define the β_i over $\alpha_1, \ldots, \alpha_n$ there exists an n-dimensional smooth projective algebraic variety S, defined $/\mathbb{Q}$ up to birational equivalence, with function field

$$\mathbb{Q}(S) \cong k.$$

- We may think of $X^* = \mathbb{P}^2 \setminus T$ and D as algebro-geometric objects defined respectively over $\mathbb{Q}$ and over the extension field k of $\mathbb{Q}$ — then S may be thought of as geometric realizations of the different embeddings $k \hookrightarrow \mathbb{C}$.
- For each $s \in S$ we have $x_i(s), y_i(s)$ and

$$D_s = \sum_i n_i(x_i(s), y_i(s))$$

  satisfies
  - $\deg D_s = 0$
  - $\prod_i x_i(s)^{n_i} = \prod y_i(s)^{n_i} = 1.$

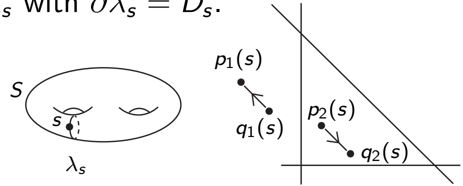
The second equation above is because any algebraic relation $/\mathbb{Q}$ satisfied by the original x_i, y_i is still satisfied for the $x_i(s), y_i(s)$.

We want to define

$$\mathrm{AJ}_2(D)$$

using $\omega = \frac{dx}{x} \wedge \frac{dy}{y}$. For this we need something real 2-dimensional to integrate ω over. For $\gamma \in H_1(S, \mathbb{Z})$ each point $s \in \gamma$ gives

- $D_s = \sum n_i(x_i(s), y_i(s)) = \Sigma$
- 1-chain λ_s with $\partial\lambda_s = D_s$.



The locus

$$\Gamma = \bigcup_{s \in \gamma} \lambda_s$$

is then of 2 *real dimensions*, and we set

$$\mathrm{AJ}_2(D) = \int_\Gamma \omega \qquad \left\{ \begin{array}{c} \text{modulo} \\ \text{ambiguities} \end{array} \right\}.$$

Using the assumption $\mathrm{AJ}_1(D_s) = 0$ the ambiguities can be made sense of.

One should think of $\mathrm{AJ}_2(D)$ as involving one integration in a geometric direction and one integration in an arithmetic direction. This is the new, additional ingredient that appears in Hodge theory when studying algebraic cycles of codimension $\geqq 2$.

What so far as I know has not been done is to show that

$$D \sim 0 \iff \mathrm{AJ}_i(D) = 0 \text{ for } i = 0, 1, 2.$$

The implication $\implies$ is OK;[6] missing is an interpretation

$$\mathrm{AJ}_2(D) \in K_2(\mathbb{C})$$

and an argument that

$$\mathrm{AJ}_2(D) = 0 \implies D \sim 0 \quad \text{(mod torsion)}$$

This would be the full converse to Abel's theorem for this example.

[6]That is, $D \sim 0 \implies \mathrm{AJ}_2(D) \equiv 0 \bmod \{\text{periods} + \text{ambiguities}\}$.

Conclusion: The HC is formulated for smooth complex algebraic varieties. A proof requires that we construct algebraic subvarieties starting from a homology class that satisfies Hodge-theoretic conditions. However there are Hodge-theoretic invariants of an algebraic cycle that arise arithmetically, and a deeper understanding of these may be necessary for HC. Basically we have to relate the arithmetic and geometric properties of periods.