Homotopy Theory of Smooth Algebraic Varieties

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Additional References for this lecture

J. Carlson, H. Clemens, and J. Morgan, "On the mixed Hodge structure associated to π_3 of a simply connected projective manifold," *Ann. Sci. ENS*, **14** (1981), 323 – 338.

P. Deligne, "Théorie de Hodge,II," *Publ. IHES*, **40** (1971), 5 – 58.

J. Morgan, "The algebraic topology of smooth algebraic varieties, *Publ. IHES*, **48** (1978), 137 – 204.

A. Weil, "Introduction à l'étude des varietiés kahlériennes," Paris, Hermann, 1958.

Today's talk we focus on Hodge structures in the homotopy theory of compact Kahler manifolds and open smooth complex algebraic varieties.

- Brief review of analysis on compact Kahler manifolds
- Definitions of Hodge structure and Mixed Hodge structures.
- Formality for compact Kahler manifolds
- Principle of two types and mixed Hodge structures in homotopy theory of compact Kahler manifolds
- An example: MHS on π_3
- The case of open, smooth complex algebraic varieties

PART I: BRIEF REVIEW OF KAHLER GEOMETRY

Kahler Metrics

A Hermitian metric on a complex manifold is given by

$$\langle \xi, \eta \rangle = S(\xi, \eta) + iA(\xi, \eta)$$

with S a Riemannian metric and A a two-form, called the Kahler form. We also have $A(\xi, \eta) = S(\xi, J\eta)$. The metric is *Kahler* if A is closed. Equivalently, the Hermitian metric is Kahler if J is parallel under the Levi-Civita connection of the Riemannian metric S, or what amounts to the same thing, there are local holomorphic coordinates centered at each point in which the metric at the point is standard to second order.

The $\partial\overline{\partial}$-lemma

The Kahler condition implies graded commutation results

$$[\partial, \overline{\partial}^*] = [\partial^*, \overline{\partial}] = 0,$$

and the close relationship of the Laplacians:

$$\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d.$$

These local equations have strong analytic consequences when the manifold is compact, namely the $\partial\overline{\partial}$-lemma:

Lemma

($\partial\overline{\partial}$-lemma) Suppose that α is form closed under both ∂ and $\overline{\partial}$ (and hence also closed under d). Then if it is exact under any of the three operators there is β such that

$$\partial\overline{\partial}\beta = \alpha.$$

There is the equivalent dd^c-lemma which says that under the

Hodge Theory for compact Kahler manifolds

The complex structure gives us a decomposition of the differential forms into types: $\Omega^k(M;\mathbb{C}) = \oplus_{p+q=k}\Omega^{p,q}(M)$, the latter being forms that in local holomorphic coordinates are a wedge product of a C^∞-function, p holomorphic one-forms and q anti-holomorphic one-forms. There is the associated decreasing filtration $F^r(\Omega^*(M;\mathbb{C}) = \oplus_{(p,q)|p\geq r}\Omega^{p,q}(M)$.

There are two immediate consequences of the $\partial\overline{\partial}$-lemma:

- The spaces of d-harmonic, ∂-harmonic, and $\overline{\partial}$-harmonic forms are all the same.
- The Hodge to deRham spectral sequence degenerates at E_1.

The first statement tells us that $H^k(M;\mathbb{C}) = \oplus_{p+q=k}H^{p,q}$, where $H^{p,q}$ is the space of harmonic forms of type (p,q), and $\overline{H^{p,q}} = H^{q,p}$. A priori, this decomposition may depend on the metric, but in fact it does not since the second implies that $H^{p,q} = F^q(H^{p+q}) \cap \overline{F}^q(H^{p+q})$ where $F^*(H^{p+q})$ is the filtration induced from the Hodge filtration on the differential forms.

PART II: HODGE AND MIXED HODGE STRUCTURES

Definition

A $\mathbb{Q}$-Hodge structure of weight k on a rational vector space V is decomposition $V \otimes \mathbb{C} = \oplus_{p+q=k} V^{p,q}$ with $V^{q,p} = \overline{V^{p,q}}$. Or equivalently, it is a decreasing filtration F^* on $V \otimes \mathbb{C}$ with the property that for all $p + q = k$ we have $F^p \oplus \overline{F^{q+1}} = V \otimes \mathbb{C}$.

One can also define an integral Hodge structure and a Hodge structure with respect to any subfield of $\mathbb{R}$.

Definition

A rational mixed Hodge structure consists of:

- *a rational vector space* V,
- *an increasing filtration* $W_*(V)$, *called the* weight filtration, *and*
- *a decreasing filtration* $F^*(V \otimes \mathbb{C})$ *called the* Hodge filtration.

These are required to satisfy the condition that for each k *the filtration on* W_k/W_{k-1} *induced from* F^* *gives an ordinary Hodge structure of weight* k.

Example. If $W_{k-1} = 0$ and $W_k = V$ then we have an ordinary Hodge structure of weight k on V.

A non-trivial mixed Hodge structure

Consider an integral mixed Hodge structure of Hodge-Tate type written as an extension:

$$0 \to W_{-2} \subset W_0 = H_{\mathbb{Z}}$$

where $W_{-2} = (2\pi i)\mathbb{Z}$ and $W_0/W_{-2} = \mathbb{Z}$. The Hodge structure on W_{-2} of type $(-1,-1)$ and that on W_0/W_{-2} is of type $(0,0)$. Such extensions are classified by $q \in \mathbb{C}^*$ as follows: Let $(2\pi i)e_1 \in H_{\mathbb{Z}}$ be the image of $2\pi i \in W_{-2}$. Let $e_0 \in F^0(W_0)$ be the pre-image of $1 \in W_0/W_{-2}$ and let $e_0' \in H_{\mathbb{Z}}$ map to $1 \in W_0/W_{-2}$. Then $e_0 = e_0' + \tau e_1$. The only indeteminacy is in the choice of e_0' which can be varied by an integral multiple of $(2\pi i)e_1$, and this changes τ by $(2\pi i)$ times an integer. Thus, the invariant $q \in \mathbb{C}^*$ is $\exp(\tau)$, or said another way $\tau = \log(q)$, with the choice of branch of the log being irrelevant.

PART III: FORMALITY

Formal DGAs

The cohomology of a DGA is itself a DGA, one with trivial d.

Definition

A DGA is formal *if it is homotopy equivalent from it to its cohomology by an equivalence inducing the identity on cohomology.*

Theorem

Let A^* be a $\mathbb{Q}$-DGA. If $A^* \otimes \mathbb{C}$ is formal then so is A^*.

The idea is that formality is equivalent to an action of the multiplicative group G_m on the DGA so that the induced action on cohomology is given by multiplying by the k^{th} power on H^k. When the DGA is defined over K having such an action over the algebraic closure is equivalent to having one defined over K.

Formal DGAs

Examples.

- $\Omega^*(\mathbb{C}P^n)$ is formal.
- For M a hypersurface in $\mathbb{C}P^n$, $\Omega^*(M)$ is formal.
- If M is formal then all Massey triple products $\langle a, b, c\rangle$ vanish.
- The nilpotent completion of a finitely generated free group is formal.
- Conversely, if the nilpotent completion of a finitely generated group G is formal and if the cup product $H^1(BG) \otimes H^1(BG) \to H^2(BG)$ is trivial, then this nilpotent completion is that of a free group.

Massey triple products

Suppose a, b, c are cohomology classes with $ab = bc = 0$. Take closed form representatives α, β, γ. Then there are η with $d\eta = \alpha \wedge \beta$ and λ with $d\lambda = \beta \wedge \gamma$. The Massey triple product $\langle a, b, c\rangle$ is represented by the closed form $\eta \wedge \gamma - (-1)^{|a|}\alpha \wedge \lambda$. The class is well-defined modulo the ideal in the cohomology ring generated by a and c.

Formality for compact Kahler Manifolds

We have the following theorem proved by Deligne-Griffiths-Morgan-Sullivan

Theorem

Let M be a compact Kalher manifold. The complex deRham complex $\Omega^*(M;\mathbb{C})$ is formal.

The quickest proof is that the $\partial\overline{\partial}$-lemma shows that both maps in the following diagram of DGAs are homotopy equivalences:

$$\begin{tikzcd} \Omega^*(M;\mathbb{C}) & (\mathrm{Ker}(\overline{\partial}), \partial) \arrow[l, hook', "\text{incl.}"'] \arrow[d, "\text{projection}"] \\ & H^*(M;\mathbb{C}) \end{tikzcd}$$

Formality of Kahler Manifolds

The theorem about the formality of the differential forms on a Kahler manifold M, then implies that the rational pl forms on any triangulation of M is formal, which in turn by the main results in yesterday's lecture imply that the rational Postinov tower of M is determined by its rational cohomology ring. In particular, the rational homotopy groups of M and the rational k-invariants of its Postnokov tower can be read off from the rational cohomology ring.

PART IV: PRINCIPLE OF TWO TYPES AND MIXED HODGE STRUCTURES IN HOMOTOPY THEORY

Principle of Two Types

Theorem

Let M be a compact Kahler manifold. There is a minimal DGA $\mathcal{M}$ with a decomposition $\mathcal{M} = \oplus_{i,j}\mathcal{M}^{i,j}$ (invariant under the $\mathbb{C}^*$ action that acts by the k^{th} power on degree k) so that d and $\wedge$ are homogeneous of degree $(0,0)$. Furthermore, if $\alpha \in \mathcal{M}^{i,j}$ then $\deg(\alpha) \leq i+j$. Lastly, there are homotopy equivalences that are homotopic

$$\rho\colon \mathcal{M} \to \Omega^*(M;\mathbb{C}) \quad \text{and} \quad \rho'\colon \mathcal{M} \to \Omega^*(M;\mathbb{C})$$

such that $\rho(\mathcal{M}^{i,j}) \subset F^i(\Omega^*(M;\mathbb{C})$ and $\rho'(\mathcal{M}^{i,j}) \subset \overline{F}^j(\Omega^*(M;\mathbb{C}))$.

Consequence of the Principle of Two Types

Since d is homoegeous of bidegree $(0,0)$, $H^*\mathcal{M} = \oplus_{i,j} H^*(\mathcal{M}^{i,j})$. Suppose that $\alpha \in \mathcal{M}^{i,j}$ is closed and that the degree of α is less than $i+j$. Then $\rho(\alpha) \in F^j$ and $\rho'(\alpha) \in \overline{F}^j$ and these closed classes are cohomologous. It follows that $[\rho(\alpha)] = 0$ in $H^*(M;\mathbb{C})$ and since ρ is a homotopy equivalence α is exact.
Let $\mathcal{M}'$ be the subspace of $\mathcal{M}$ consisting of the sum over all i,j of the intersection of $\mathcal{M}^{i,j}$ with the subspace of elements of degree $i+j$. Clearly, $\mathcal{M}'$ is a sub-DGA of $\mathcal{M}$ and $d=0$ on $\mathcal{M}'$. Furthermore, the above shows that the induced map on cohomology is an isomorphism. This gives a homotopy equivalence between $\mathcal{M}$ and its cohomology, giving another proof of formality.

Another Consequence of the Principle of Two Types

We define a Hodge filtration $F^k(\mathcal{M}) = \sum_{i \geq k} \mathcal{M}^{i,j}$ and a weight filtrations $W_k(\mathcal{M}) = \sum_{i+j \leq k} \mathcal{M}^{i,j}$. The weight filtration in fact exists on the minimal model $\mathcal{M}_{\mathbb{Q}}$ of the pl forms on some smooth triangulation of M. The differential of $\mathcal{M}_{\mathbb{Q}}$ is strictly compatible with the rational weight filtration. Furthermore, there is a filtered isomorphism $(\mathcal{M}_{\mathbb{Q}}, W_*) \otimes \mathbb{C} \to (\mathcal{M}, W_*)$. Thus, $\mathcal{M}$ has a Hodge filtration, a rational structure and a rational weight filtration. These define a mixed Hodge structure on $\mathcal{M}$. This mixed Hodge structure depends on the isomorphism between the rational minimal model tensored with $\mathbb{C}$ and the complex minimal model, and that identification is only well-defined up to homotopy. Thus, the mixed Hodge structure on the minimal model is only unique up to homotopy.

Mixed Hodge structures on homotopy groups

Any rigid invariant derived from the minimal model will then inherit a well-defined mixed Hodge structure. For example, in the simply connected case the homotopy groups have well-defined integral mixed Hodge structures, as do the cohomology algebras of the various stages in the Postnikov tower. In the non-simply connected case the tower of rational Lie algebras of the tensor product with $\mathbb{Q}$ of the nilpotent quotients of the fundamental group have compatible rational mixed Hodge structures.

PART V: AN EXAMPLE WITH π_3

Example of the MHS on π_3

Let me describe a result worked out with Jim Carlson and Herb Clemens.
Let X be a simply connected compact Kahler manifold X of complex dimension N. There is a short exact sequence

$$0 \to H^3(X;\mathbb{Q}) \to \mathrm{Hom}(\pi_3(X),\mathbb{Q}) \to \mathrm{Ker} \to 0$$

where $\mathrm{Ker} = \mathrm{Ker}\big(\mathrm{Sym}^2(H^2(X;\mathbb{Q})) \to H^4(X;\mathbb{Q})\big)$. This is an exact sequence of mixed Hodge structures, so that the mixed Hodge structure on $\mathrm{Hom}(\pi_3,\mathbb{Z})$ has weights 3 and 4. We examine a piece of this MHS.

Example of the MHS on π_3

Let $H^{1,1}_{\mathbb{Z}}(X) = H^{1,1}(X) \cap H^2(X;\mathbb{Z})$. Suppose that

$$\mathrm{Sym}^2(H^{1,1}_{Z}) \to H^{2,2}(X;\mathbb{C})$$

has a non-trivial kernel. For any element k in the kernel of this map we lift k to an integral vector $k_{\mathbb{Z}} \in \mathrm{Hom}(\pi_3(X),\mathbb{Z})$ and we lift k to a vector $k_2 \in F^2\big(\mathrm{Hom}(\pi_3,\mathbb{Z}) \otimes \mathbb{C}\big)$. The difference of these two lifts is an element in $W_3 \otimes \mathbb{C} = H^3(X;\mathbb{C})$ and is well-defined modulo the image of $H^3(X;\mathbb{Z}) + F^2H^3(Z;\mathbb{C})$. The image of the difference $k_{\mathbb{Z}} - k_2$ in the quotient is part of the extension date of the exact sequence of MHSs.

Example of the MHS on π_3

We write k as Poincaré dual to a linear combination of divisors with integer coefficients $\sum_{i,j} a_{i,j} D_j \cap D_j$. The fact that this linear combination vanishes in integral homology implies that there is a $2N-3$ chain γ with $\partial\gamma = \sum_{i,j} a_{i,j} D_i \cap D_j$. We consider the map $F^{N-1}(H^{2N-3}) \to \mathbb{C}$ given by integration over γ. This is a well defined element (independent of the choice of γ) of Griffiths intermediate Jacobian

$$\left(F^{N-1}(H^{2N-3})\right)^* / (\text{integral periods}).$$

Wedge product followed by integration over the top cycle induces an isomorphism $\left(H^3(X;\mathbb{C})/F^2\right) \to F^{N-1}(H^{2N-3})^*$. Because the wedge product of $\eta \in F^2(\Omega^*)$ with any form on $F^{N-1}(\Omega)$ is zero, and the element determined by integration over γ modulo periods is identified with the extension class for the MHS associated to $\sum_{i,j} a_{i,j} D_i \cap D_j$.

PART VI: OPEN SMOOTH COMPLEX ALGEBRIAC VARIETIES

Mixed Hodge Structures

Deligne proved that the cohomology of a smooth open complex algebraic variety has a mixed Hodge structure. By Hironaka's resolution of singularities any such variety is the complement of divisor with normal crossing in a smooth complete algebraic variety. Associated to this picture one was the log complex: a complex of sheaves of differential forms which are modules over the local holomorphic functions on the compactification. The forms are allowed to have logarithmic poles along the divisor D. At points where $D_1, \ldots, D_k$ meet the singularities are modeled on

$$\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_k}{z_k}.$$

This complex computes the cohomology of the complement of D and has a natural Hodge and weight filtrations. The induced filtrations on cohomology define a mixed Hodge structure with respect to the usual rational (or integral) structure. The non-trivial weights on H^n run from n to $2n$.

Extension of the principle of two types

Using the log complex and its conjugate one shows that the complex minimal model of an open smooth variety has a decomposition

$$\mathcal{M} = \oplus \mathcal{M}^{i,j}.$$

The degree of any $\alpha^{i,j}$ at most $i+j$ and both d and wedge product are homogeneous of degree $(0,0)$. The weight filtration is given by $i+j$ and it is isomorphic to the tensor product with $\mathbb{C}$ of a filration on the rational minimal model. The Hodge filtration is given by i and on W_n/W_{n-1} it is n-opposed to its conjugate. Thus, this direct sum decomposition determines a mixed Hodge structure on the minimal model. As before it is well-defined only up to homotopy.

As before any rigid homotopy invariant, for example the homotopy groups in the simply connected case, have well-defined and functorial mixed Hodge structure. Other examples are the cohomology of the various stages of the Postnokov tower. The k-invariants viewed as maps from the dual of π_n to the cohomology of the $(n-1)^{st}$ stage is a morphism of mixed Hodge structres.

A consequence for the fundamental group

The tower of rational nilpotent Lie algebras associated with the nilpotent completion of the fundamental group is a tower of Lie algebras with mixed Hodge structures. In particular, there is an algebraic action of $\mathbb{Q}^*$ on this tower that has negative weights, with all the weights -1 and -2 on the first cohomology and $-2, -3, -4$ on the second cohomology.

As one purely group theoretic consequence for fundamental groups of open smooth varieties we have the following.

Theorem

If the 4-stage of the nilpotent tower of the fundamental group of a smooth open variety is isomoprhic to the 4-stage of the nilpotent tower for a free group, then the same is true for every stage in the nilpotent tower.